

# ON THE GALOIS GROUPS OF THE DUALIZING COVERINGS FOR PLANE CURVES

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**ABSTRACT.** Let  $C_1$  be an irreducible component of a reduced projective curve  $C \subset \mathbb{P}^2$  defined over the field  $\mathbb{C}$ ,  $\deg C_1 \geq 2$ , and let  $T$  be the set of lines  $l \subset \mathbb{P}^2$  meeting  $C$  transversally. In the article, we prove that for a line  $l_0 \in T$  and any two points  $P_1, P_2 \in C_1 \cap l_0$  there is a loop  $l_t \subset T$ ,  $t \in [0, 1]$ , such that the movement of the line  $l_0$  along the loop  $l_t$  induces the transposition of the points  $P_1, P_2$  and the identity permutation of the other points of  $C \cap l_0$ .

## INTRODUCTION

Let  $C_i \subset \mathbb{P}^2$ ,  $1 \leq i \leq k$ , be irreducible reduced curves defined over the field  $\mathbb{C}$ ,  $\deg C_i = d_i \geq 2$ , and  $C = C_1 \cup \dots \cup C_k$ ,  $d = \deg C = d_1 + \dots + d_k$ . Denote by  $\nu : \overline{C} \rightarrow C$  the normalization of the curve  $C$  and consider a point  $p \in \overline{C}$  and its image  $P = \nu(p) \in \mathbb{P}^2$ . Choose homogeneous coordinates  $(x_1, x_2, x_3)$  in  $\mathbb{P}^2$  such that  $P = (0, 0, 1)$ . We can choose a local parameter  $t$  in a complex analytic neighborhood  $U \subset \overline{C}$  of the point  $p$  such that the regular map  $\nu$  is given by

$$x_1 = \sum_{i=s_p}^{\infty} a_i t^i, \quad x_2 = t^{r_p}, \quad x_3 = 1, \quad (1)$$

where  $a_{s_p} \neq 0$  and  $s_p > r_p \geq 1$ . The integer  $r_p$  is called the *multiplicity* of the germ  $\nu(U)$  of the curve  $C$  at  $P = \nu(p)$ , the line  $l_p = \{x_1 = 0\}$  is called a *tangent line* to  $C$  at  $P$ , and the integer  $s_p$  is called the *tangent multiplicity* of the germ  $\nu(U)$  at  $P$ .

Let  $\hat{C} \subset \hat{\mathbb{P}}^2$  be the dual curve to the curve  $C$  (the curve  $\hat{C}$  consists of the tangents  $l_p$ ,  $p \in \overline{C}$ , to  $C$ ). The graph of the correspondence between  $C$  and  $\hat{C}$  is a curve  $\check{C}$  (the so called Nash blow-up of  $C$ ) in  $\mathbb{P}^2 \times \hat{\mathbb{P}}^2$  which lies in the incidence variety  $I = \{(P, l) \in \mathbb{P}^2 \times \hat{\mathbb{P}}^2 \mid P \in l\}$ ,

$$\check{C} = \{(\nu(p), l_p) \in I \mid p \in \overline{C} \text{ and } l_p \text{ is the tangent line to } C \text{ at } \nu(p) \in C\}.$$

In the sequel, by  $L_p \subset \hat{\mathbb{P}}^2$ , we denote the line dual to the point  $\nu(p) \in C \subset \mathbb{P}^2$ .

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Let  $\text{pr}_1 : \mathbb{P}^2 \times \hat{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$  and  $\text{pr}_2 : \mathbb{P}^2 \times \hat{\mathbb{P}}^2 \rightarrow \hat{\mathbb{P}}^2$  be the projections to the factors,  $X = \text{pr}_1^{-1}(C) \cap I$ , and  $f' : X \rightarrow \hat{\mathbb{P}}^2$  the restriction of  $\text{pr}_2$  to  $X$ . Obviously,  $f'^{-1}(l)$  consists of the points  $(P, l) \in \mathbb{P}^2 \times \hat{\mathbb{P}}^2$  such that  $P \in C \cap l$  and hence  $\deg f' = \deg C = d$ .

Denote by  $\nu' : Z \rightarrow X$  the normalization of  $X$  and by  $f = f' \circ \nu' : Z \rightarrow \hat{\mathbb{P}}^2$ . We have  $\deg f = d$ . We call  $f$  the *dualizing covering* for  $C \subset \mathbb{P}^2$ . Obviously, the variety  $Z$  is isomorphic to the fibre product  $\overline{C} \times_C X$  of the normalization  $\nu : \overline{C} \rightarrow C$  and the projection  $\text{pr}_1 : X \rightarrow C$ . The projection  $\text{pr}_1 : X \rightarrow C$  gives on  $X$  a structure of a ruled surface and it induces a ruled structure on  $Z$  over the curve  $\overline{C}$ ,  $\rho : Z \rightarrow \overline{C}$ ,  $\rho^{-1}(p) := F_p \simeq \mathbb{P}^1$  for  $p \in \overline{C}$ . Note that the curve  $\tilde{C}$  is a section of this ruled structure, where  $\tilde{C} = \nu'^{-1}(\check{C}) \subset Z$ , and the image  $f(F_p)$  of a fibre  $F_p$  is the line  $L_p \subset \hat{\mathbb{P}}^2$  dual to the point  $\nu(p) \in C \subset \mathbb{P}^2$ .

Denote by  $\overline{B} \subset \hat{\mathbb{P}}^2$  the branch locus of  $f$ , choose a point  $l_0 \in \hat{\mathbb{P}}^2 \setminus \overline{B}$ , and number the points of  $f^{-1}(l_0)$ . In this case the covering  $f$  induces a homomorphism  $f_* : \pi_1(\hat{\mathbb{P}}^2 \setminus \overline{B}, l_0) \rightarrow \Sigma_d$  from the fundamental group  $\pi_1(\hat{\mathbb{P}}^2 \setminus \overline{B}, l_0)$  to the symmetric group  $\Sigma_d$  acting on the fibre  $f^{-1}(l_0)$ . The image,  $\text{Im} f_* := G \subset \Sigma_d$ , is called the *Galois group* of the covering  $f$ .

**Theorem 1.** *Let  $f : Z \rightarrow \hat{\mathbb{P}}^2$  be the dualizing covering for a reduced curve  $C \subset \mathbb{P}^2$ ,  $\deg C = d$ , and  $C_1, \dots, C_k$  the irreducible components of  $C$ ,  $\deg C_i = d_i \geq 2$ . Then the Galois group of  $f$  is  $G \simeq \Sigma_{d_1} \times \dots \times \Sigma_{d_k}$ .*

The following theorem describes properties of dualizing coverings.

**Theorem 2.** *Let  $C$  be as in Theorem 1 and  $f : Z \rightarrow \hat{\mathbb{P}}^2$  the dualizing covering for  $C$ . Then  $Z$  is a non-singular surface consisting of  $k$  irreducible components and  $f$  is a degree  $d$  finite covering.*

*The branch locus of  $f$  is  $\overline{B} = \hat{C} \cup \hat{L}$ , where  $\hat{L} = \bigcup_{r_p \geq 2} L_p$ ,  $L_p$  are the lines*

*dual to the points  $\nu(p) \in C$  and the union is taken over all  $p \in \overline{C}$  for which the multiplicity  $r_p \geq 2$ .*

*The ramification locus of  $f$  is  $\overline{R} = \tilde{C} \cup \tilde{F}$ , where  $\tilde{F} = \bigcup_{r_p \geq 2} F_p$  and the union*

*is taken over all  $p \in \overline{C}$  for which  $r_p \geq 2$ .*

*The local degree  $\deg_q f$  of  $f$  at a point  $q = F_p \cap \tilde{C}$  is equal to the tangent multiplicity  $s_p$ , and  $\deg_q f = r_p$  at all points  $q \in F_p \setminus \tilde{C}$ . For all points  $q \in \tilde{C} \setminus \tilde{F}$  the local degree  $\deg_q f = 2$ .*

For given reduced projective curve  $C \subset \mathbb{P}^2$ ,  $\deg C = n$ , let  $T_C$  be the set of lines  $l \subset \mathbb{P}^2$  meeting  $C$  transversally. Let  $l_t \subset T_C$ ,  $t \in [0, 1]$ , be a loop and let  $l_0 \cap C = \{P_1, \dots, P_n\}$ . Then the movement of the line  $l_0$  along the loop  $l_t$  defines  $n$  paths  $\psi_i(t) = l_t \cap C \subset \mathbb{P}^2$ ,  $i = 1, \dots, n$ , starting and ending at the points

$P_1, \dots, P_n$  and, consequently, induces a permutation of the points  $P_1, \dots, P_n$  called the *monodromy* of the points  $P_1, \dots, P_n$  along the loop  $l_t$  (the start point  $P_i = \psi_i(0)$  of the path  $\psi_i(t)$  maps to the end point  $\psi_i(1) \in l_0 \cap C$ ).

**Corollary 1.** *Let  $C_1$ ,  $\deg C_1 \geq 2$ , be an irreducible component of a reduced curve  $C \subset \mathbb{P}^2$ . For a line  $l_0 \subset T_C$  and any two points  $P_1, P_2 \in C_1 \cap l_0$  there is a loop  $l_t \subset T_C$ ,  $t \in [0, 1]$ , such that the monodromy along the loop  $l_t$  is the transposition of the points  $P_1, P_2$  and the identity permutation of the other points in  $C \cap l_0$ .*

The proof of Theorems 1, 2 and Corollary 1 will be given in Section 3. In Section 1, we give some remarks on the actions of finite groups on finite sets and prove Proposition 1 which plays the crucial role in the computation of the Galois groups of the dualizing coverings in the proof of Theorem 1. In Section 2, we remind some properties of finite ramified coverings and begin to prove Theorem 1.

For background results on plane algebraic curves and dual to them we refer to [1] and [3].

## 1. SOME REMARKS ON THE ACTIONS OF FINITE GROUPS

Let  $I$  be a finite set and  $\Sigma_I$  the symmetric group acting on  $I$ . An embedding  $I_1 \subset I_2$  defines the natural embedding  $\Sigma_{I_1} \subset \Sigma_{I_2}$ . In the sequel, we will assume that each finite set  $I$  is a subset of an integer segment  $[1, d] = \{1, 2, \dots, d\}$ , so that  $\Sigma_I \subset \Sigma_{[1, d]} := \Sigma_d$ .

Let  $J = \{I_1, \dots, I_k\}$  be a partition of the segment  $[1, d]$ . The partition  $J$  defines an embedding of the group  $\Sigma_J := \Sigma_{I_1} \times \dots \times \Sigma_{I_k}$  into  $\Sigma_d$ .

We say that a partition  $J$  of  $[1, d]$  is *invariant* under the action of a subgroup  $G \subset \Sigma_d$  if  $g(I_j) = I_j$  for all  $g \in G$  and all  $I_j \in J$ .

Let  $J_i = \{I_{i,1}, \dots, I_{i,k_i}\}$ ,  $i = 1, 2$ , be two partitions of  $[1, d]$ . We say that the partition  $J_1$  is *thinner* than  $J_2$  (resp.,  $J_2$  is *thicker* than  $J_1$ ) and write  $J_1 \preceq J_2$  if for each  $j$ ,  $1 \leq j \leq k_1$ , there is  $t(j)$  such that  $I_{1,j} \subset I_{2,t(j)}$ . For any two partitions  $J_i = \{I_{i,1}, \dots, I_{i,k_i}\}$ ,  $i = 1, 2$ , denote by  $J_1 \oplus J_2$  the thinnest partition of  $[1, d]$  such that  $J_1 \preceq J_1 \oplus J_2$  and  $J_2 \preceq J_1 \oplus J_2$ .

**Claim 1.** *Let  $G$  be a subgroup of  $\Sigma_d$  and let  $J_i = \{I_{i,1}, \dots, I_{i,k_i}\}$ ,  $i = 1, 2$ , be two partitions of  $[1, d]$  such that  $\Sigma_{J_i} \subset G$  for  $i = 1, 2$ . Then  $\Sigma_{J_1 \oplus J_2} \subset G$ .*

*Proof.* Obvious. □

It follows from Claim 1 that for each subgroup  $G$  of  $\Sigma_d$  there is the thickest partition of  $[1, d]$  (denote it by  $J_G$ ) such that  $\Sigma_{J_G} \subset G$ .

Let  $\sigma = c_1 \cdot \dots \cdot c_n \in \Sigma_d$  be the factorization of  $\sigma$  into the product of cycles with disjoint orbits. The number  $n_\sigma = n$  will be called the *length* of cycle factorization.

**Lemma 1.** *Let  $H$  be a subgroup of  $\Sigma_d$  generated by a set of transpositions and a permutation  $\sigma$ , and let  $\sigma = c_1 \cdot \dots \cdot c_{n_\sigma}$  be the factorization of  $\sigma$  into the product of cycles with disjoint orbits. Assume that for each  $i$ ,  $1 \leq i \leq n_\sigma$ , there is a partition  $J_i = \{I_{i,1}, \dots, I_{i,k_i}\}$  of  $[1, d]$  invariant under the action of  $\sigma$  and such that*

- (i) *for each  $I_{i,j} \in J_i$  there is at most one cycle  $c_{m(i,j)}$  entering into the factorization of  $\sigma$  such that the cycle  $c_{m(i,j)}$  acts non-trivially on  $I_{i,j}$ ,*
- (ii) *the cycle  $c_i$  acts non-trivially on  $I_{i,1}$  and the length of the cycle  $c_i$  is strictly less than the cardinality of  $I_{i,1}$ ,*
- (iii) *the group  $H$  acts transitively on  $I_{i,1}$ .*

*Then  $H = \Sigma_{J_H}$  and, in particular,  $H$  is generated by transpositions.*

*Proof.* Consider a set  $I_{i,1}$ . Let  $l_{i,1}$  be its cardinality and let  $l_i$  be the length of the cycle  $c_i$ . We have  $l_i < l_{i,1}$  and it follows from (i) and (iii) that there exists a transposition  $\tau \in H \cap \Sigma_{I_{i,1}}$  such that it commutes with  $c_j$  if  $j \neq i$  and it transposes an element entering in the cycle  $c_i$  and an element of  $I_{i,1}$  which does not enter in  $c_i$ . Without loss of generality, we can assume that  $I_{i,1} = \{1, 2, \dots, l_{i,1}\}$ ,  $c_i = (1, 2, \dots, l_i)$ , and  $\tau = (l_i, l_i + 1)$ . Therefore

$$\sigma^{-j} \tau \sigma^j = (l_i - j, l_i + 1) \in H$$

for  $j = 1, \dots, l_i - 1$  and hence  $H \cap \Sigma_{l_i+1} = \Sigma_{l_i+1}$ , since the subgroup of  $H$ , generated by the transpositions  $\sigma^{-j} \tau \sigma^j$ ,  $j = 0, 1, \dots, l_i - 1$ , acts transitively on the set  $\{1, 2, \dots, l_i + 1\}$ . If we apply conditions (i) – (iii)  $l$  times, where  $l = l_{i,1} - l_i$ , we obtain that  $\Sigma_{I_{i,1}} \subset H$  for each  $i$  and hence  $\sigma$  is a product of some transpositions belonging to  $H$ .  $\square$

The following proposition is an easy consequence of Claim 1 and Lemma 1.

**Proposition 1.** *Let  $G$  be a subgroup of  $\Sigma_d$  generated by some set of transpositions and by permutations  $\sigma_1, \dots, \sigma_m$ . Assume that for each  $i$ ,  $i = 1, \dots, m$ , there are partitions  $J_{i,j}$  of  $[1, d]$ ,  $1 \leq j \leq n_{\sigma_i}$ , such that the subgroup  $H_i$  of  $G$ , generated by transpositions and by  $\sigma_i$ , and the partitions  $J_{i,j}$  satisfy the conditions of Lemma 1. Then  $G = \Sigma_{J_G}$ .*

**Corollary 2.** *Let  $G \subset \Sigma_d$  satisfy the conditions of Proposition 1. Assume that there is a partition  $J = \{I_1, \dots, I_k\}$  of  $[1, d]$  such that  $G$  leaves invariant the partition  $J$  and acts transitively on each  $I_j \in J$ ,  $1 \leq j \leq k$ . Then  $J = J_G$  and  $G = \Sigma_J$ .*

## 2. COVERINGS

By a *covering* we understand a branched covering, that is a finite morphism  $f : Z \rightarrow Y$  from a normal projective surface  $Z$  onto a non-singular irreducible projective surface  $Y$ . To each covering  $f$  we associate the branch locus  $\overline{B} \subset Y$ , the ramification locus  $\overline{R} \subset f^{-1}(\overline{B}) \subset Z$ , and the unramified part  $Z \setminus f^{-1}(\overline{B}) \rightarrow Y \setminus \overline{B}$  (which is the maximal unramified subcovering). As is usual for unramified coverings of degree  $d$ , there is a homomorphism  $f_*$  which acts from the fundamental group  $\pi_1(Y \setminus \overline{B}, p_0)$  to the symmetric group  $\Sigma_d$  acting on the points of  $f^{-1}(p_0)$ . The homomorphism  $f_*$  (called *monodromy* of  $f$ ) is defined by  $f$  uniquely if we number the points of  $f^{-1}(p_0)$ ; reciprocally, according to Grauert-Remmert-Riemann-Stein Extension Theorem (see, for example, [2]) the conjugacy class of  $f_*$  defines  $f$  up to an isomorphism. The image  $G \subset \Sigma_d$  of  $f_*$  is a transitive subgroup of  $\Sigma_d$  if  $Z$  is irreducible and in general case the number of connected components of  $Z$  is equal to the number of orbits of the action of  $G$  on  $f^{-1}(p_0)$ .

An element  $\gamma_q, q \in \overline{B} \setminus \text{Sing } \overline{B}$ , of the fundamental group  $\pi_1(Y \setminus \overline{B}, p_0)$  is called a *geometric generator* if it is represented by a loop  $\Gamma_q$  of the following form. To define  $\Gamma_q$ , let  $L \subset Y$  be a curve meeting  $\overline{B}$  transversely at  $q$  and let  $S^1 \subset L$  be a circle of small radius with center at  $q$ . The choice of an orientation on  $Y$  defines an orientation on  $S^1$ . Then  $\Gamma_q$  is a loop consisting of a path  $l$  in  $Y \setminus \overline{B}$  joining  $p_0$  with a point  $q_1 \in S^1$ , the loop  $S^1$  (with positive direction) starting and ending at  $q_1$ , and a return path to  $p_0$  along  $l$  in the opposite direction (of course, we must note that a geometric generator  $\gamma_q$  depends not only on  $q$ , but it depends also on the choice of the path  $l$ ). Note that if  $Y$  is simply connected then  $\pi_1(Y \setminus \overline{B}, p_0)$  is generated by geometric generators.

In the sequel, we will assume that the covering  $f$  satisfies some additional conditions. The first of them is:

- ( $R_0$ ) *If for an irreducible component  $\overline{B}_i$  of the branch curve  $\overline{B}$  the image  $f_*(\gamma_{q_i})$  of a geometric generator  $\gamma_{q_i} \in \pi_1(Y \setminus \overline{B}, p_0)$ ,  $q_i \in \overline{B}_i$ , is not a transposition, then  $\overline{B}_i$  is a smooth curve and  $f|_{\overline{R}_{i,j}} : \overline{R}_{i,j} \rightarrow \overline{B}_i$  is an isomorphism for all  $j$ ,  $1 \leq j \leq n$ , where  $\overline{R} \cap f^{-1}(\overline{B}_i) = \overline{R}_{i,1} \cup \dots \cup \overline{R}_{i,n}$  is the decomposition of  $\overline{R} \cap f^{-1}(\overline{B}_i)$  into the union of irreducible components.*

Let  $r_{i,j}$  be the ramification multiplicity of  $f$  along  $\overline{R}_{i,j}$  (that is, the local degree of  $f$  at a generic point of  $\overline{R}_{i,j}$ ), then the cycle type of the permutation  $f_*(\gamma_{q_i}) \in \Sigma_d$  is  $(r_{i,1}, \dots, r_{i,n})$ .

Let for  $\overline{B}_1$  the image  $f_*(\gamma_{q_1})$  be not a transposition and let  $\overline{R}_1, \dots, \overline{R}_n$  be the irreducible components of  $\overline{R} \cap f^{-1}(\overline{B}_1)$ . For each point  $o \in \overline{B}_1 \cap \text{Sing } \overline{B}$  let us choose a very small (in complex analytic topology) neighbourhood  $W_o \subset Y$  of

the point  $o$ . Denote by  $B_1 := \overline{B_1} \setminus (\bigcup_{o \in \text{Sing } \overline{B}} \overline{W}_o)$  and  $R_j := \overline{R_j} \cap f^{-1}(B_1)$ , where

$\overline{W}_o$  is the closure of  $W_o$  in  $Y$ . The following Lemma is well known.

**Lemma 2.** *There are neighbourhoods  $U_1 \subset Y$  and  $V_j \subset Z$ ,  $j = 1, \dots, n$ , such that*

- (i)  $U_1 \cap \overline{B} = B_1$  and  $V_j \cap \overline{R_j} = R_j$ ,
- (ii)  $U_1$  is biholomorphic to  $B_1 \times D_1$  and  $V_j$  is biholomorphic to  $R_j \times D_2$ , where  $D_1 = \{u_1 \in \mathbb{C} \mid |u_1| < 1\}$  and  $D_2 = \{v_1 \in \mathbb{C} \mid |v_1| < 1\}$  are discs in  $\mathbb{C}$ ,
- (iii) the restriction of  $f$  to each  $V_j$  is proper and  $f(V_j) = U_1$ ,
- (vi) if  $u_2$  is a local parameter on  $B_1$  at a point  $p \in B_1$  and  $v_{2,j} = f_{|R_j}^*(u_2)$  is the local parameter at the point  $q_j = f_{|R_j}^{-1}(p)$  on  $R_j$ , then  $f : V_j \rightarrow U_1$  is given by  $u_1 = v_1^{r_1}$  and  $u_2 = v_{2,j}$  in a neighbourhood of the point  $q_j$ .

Consider a neighbourhood  $U_1$  the existence of which is claimed in Lemma 2. Let  $\text{pr} : U_1 \rightarrow B_1$  be the projection defined by bi-holomorphic isomorphism  $U_1 \simeq B_1 \times D_1$ . Let us choose a point  $q_1 \in B_1$ , a point  $p_1 \in \text{pr}^{-1}(q_1) \simeq D_1$  lying in the circle  $\delta = \{q_1\} \times \{u_1 \in \mathbb{C} \mid |u_1| = \frac{1}{2}\} \subset \{q_1\} \times D_1 \subset U_1 \setminus B_1$  (let, for definiteness,  $u_1 = \frac{1}{2}$  at the point  $p_1$ ), and a path  $l_1 \subset Y \setminus \overline{B}$  connecting the points  $p_0$  and  $p_1$ . The choice of  $l_1$  defines homomorphisms  $\text{im}_{l_1} : \pi_1(U_1 \setminus B_1, p_1) \rightarrow \pi_1(Y \setminus \overline{B}, p_0)$  and  $\varphi_1 = f_* \circ \text{im}_{l_1} : \pi_1(U_1 \setminus B_1, p_1) \rightarrow \Sigma_d$ . Denote by  $H_{B_1}$  the image  $\varphi_1(\pi_1(U_1 \setminus B_1, p_1))$  in  $G$ . Let  $\gamma_{q_1} \in \pi_1(U_1 \setminus B_1, p_1)$  be a geometric generator represented by the circle  $\delta$  (the circuit in positive direction). The cycle type of the permutation  $\sigma_1 = \varphi_1(\gamma_{q_1})$  is  $(r_{1,1}, \dots, r_{1,n})$ .

Let a set  $S = \{o_1, \dots, o_m\}$  be the intersection of  $\overline{B}_1$  and  $\text{Sing } \overline{B}$ . For a point  $o_i \in S$  we choose a small (in complex analytic topology) simply connected neighbourhood  $U_{o_i} \subset Y$  of the point  $o_i$  such that the number  $k_i$  of the connected components  $V_{o_i,1}, \dots, V_{o_i,k_i}$  of  $f^{-1}(U_{o_i})$  is equal to the number of points belonging to  $f^{-1}(o_i)$ . In addition,  $U_{o_i}$  can be chosen so that  $\overline{W}_{o_i} \subset U_{o_i}$ , where  $W_{o_i}$  is the neighbourhood of  $o_i$  used in the definition of the neighbourhood  $U_1$ .

Let us choose points  $q_{o_i} \in B_1 \cap U_{o_i}$  and paths  $l'_{o_i} \subset B_1$  connecting, respectively, the point  $q_1$  with the points  $q_{o_i}$ . Let  $l_{o_i} = \{p \in U_1 \setminus B_1 \mid \text{pr}(p) \in l'_{o_i}, u_1(p) = \frac{1}{2}\}$  be paths connecting the point  $p_1$ , respectively, with points  $p_{o_i} = \text{pr}^{-1}(q_{o_i}) \cap l_{o_i} \in U_1 \cap U_{o_i}$  (without loss of generality, we can assume that  $\text{pr}^{-1}(q) \subset U_{o_i}$  if  $q \in B_1 \cap U_{o_i}$ ). Denote by  $\tilde{l}_{o_i}$  the composition of paths  $l_1$  and  $l_{o_i}$  connecting the point  $p_0$  with the point  $p_{o_i}$ .

The path  $\tilde{l}_{o_i}$  defines homomorphisms  $\text{im}_{\tilde{l}_{o_i}} : \pi_1(U_{o_i} \setminus (U_{o_i} \cap \overline{B}), p_{o_i}) \rightarrow \pi_1(Y \setminus \overline{B}, p_0)$  and  $\varphi_{\tilde{l}_{o_i}} = f_* \circ \text{im}_{\tilde{l}_{o_i}} : \pi_1(U_{o_i} \setminus (U_{o_i} \cap \overline{B}), p_{o_i}) \rightarrow \Sigma_d$ . Denote by  $H_{o_i}$  the image  $\varphi_{\tilde{l}_{o_i}}(\pi_1(U_{o_i} \setminus (U_{o_i} \cap \overline{B}), p_{o_i}))$  in  $G$ .

Similarly, if  $U = U_1 \cup (\bigcup_{o_i \in S} U_{o_i})$ , then the path  $l_1$  defines homomorphisms  $\text{im}_{l_1} : \pi_1(U \setminus (U \cap \overline{B}), p_1) \rightarrow \pi_1(Y \setminus \overline{B}, p_0)$  and  $\varphi = f_* \circ \text{im}_{l_1} : \pi_1(U \setminus (U \cap \overline{B}), p_1) \rightarrow \Sigma_d$ , and the paths  $l_{o_i}$  define homomorphisms  $\text{im}_{l_{o_i}} : \pi_1(U_{o_i} \setminus (U_{o_i} \cap \overline{B}), p_{o_i}) \rightarrow \pi_1(U \setminus (U \cap \overline{B}), p_1)$  and  $\psi_{l_{o_i}} = f_* \circ \text{im}_{l_{o_i}} : \pi_1(U_{o_i} \setminus (U_{o_i} \cap \overline{B}), p_{o_i}) \rightarrow \Sigma_d$ . Denote by  $H_{\overline{B}_1}$  the image  $\varphi(\pi_1(U \setminus (U \cap \overline{B}), p_1))$  in  $G$ . It is easy to see that  $\varphi_{\tilde{l}_{o_i}} = \psi_{l_{o_i}}$ . Therefore,  $H_{B_1} \subset H_{\overline{B}_1}$  and  $H_{o_i} \subset H_{\overline{B}_1}$ .

Let  $\gamma_{q_{o_i}} \in \pi_1(U_{o_i} \setminus (U_{o_i} \cap \overline{B}), p_{o_i})$  be a geometric generator represented by the circle  $\delta_{o_i} = \{q_{o_i}\} \times \{u_1 \in \mathbb{C} \mid |u_1| = \frac{1}{2}\} \subset \{q_{o_i}\} \times D_1 \subset U_{o_i} \setminus \overline{B}$  (the circuit in positive direction). It is easy to see that  $\text{im}_{l_{o_i}}(\gamma_{q_{o_i}}) = \gamma_{q_1}$ . Therefore  $\varphi_{\tilde{l}_{o_i}}(\gamma_{q_{o_i}}) = \sigma_1$ .

Consider the restriction of  $f$  to each  $V_{o_i, m}$ ,  $f_{i, m} = f|_{V_{o_i, m}} : V_{o_i, m} \rightarrow U_{o_i}$ . Denote by  $d_{i, m}$  the degree of  $f_{i, m}$ ,  $d = d_{i, 1} + \dots + d_{i, k_i}$ . By construction, for the point  $\bar{o}_{i, m} = V_{o_i, m} \cap f^{-1}(o_i)$  we have  $\deg_{\bar{o}_{i, m}} f = d_{i, m}$ .

The numbering of the points of  $f^{-1}(p_0)$  and the path  $\tilde{l}_{o_i}$  define a numbering of the points of  $f^{-1}(p_{o_i})$ . Then the decomposition  $f^{-1}(U_{o_i}) = V_{o_i, 1} \sqcup \dots \sqcup V_{o_i, k_i}$  defines a partition  $J_i = \{I_{i, 1}, \dots, I_{i, k_i}\}$  of  $[1, d]$ ,  $j \in I_{i, m}$  if and only if  $\tilde{p}_j \in f^{-1}(p_{o_i}) \cap V_{o_i, m}$ . By construction, the group  $H_{o_i}$  leaves invariant the partition  $J_i$  and acts transitively on each  $I_{i, m} \in J_i$ . In particular, the action of  $\sigma_1$  leaves invariant the partition  $J_i$ .

Assume that if  $\overline{B}_j$  is an irreducible component of the branch locus  $\overline{B}$  of a covering  $f$  such that  $f_*(\gamma_{q_j})$  is not a transposition, then  $f$  satisfies the following conditions:

- (R<sub>1</sub>) For each  $o_i \in \overline{B}_j \cap \text{Sing } \overline{B}$  and each  $V_{o_i, m}$  there is at most one irreducible component  $\overline{R}_k \subset f^{-1}(\overline{B}_j)$  of the ramification locus of  $f$  which intersects with  $V_{o_i, m}$ .
- (R<sub>2</sub>) For each  $\overline{R}_k \subset f^{-1}(\overline{B}_j)$  there is  $o_i \in \overline{B}_j \cap \text{Sing } \overline{B}$  and  $m$  such that  $\overline{R}_k \cap V_{o_i, m} \neq \emptyset$  and  $r_k < d_{i, m}$ .
- (R<sub>3</sub>) If  $\overline{R}_k \cap V_{o_i, m} \neq \emptyset$  and  $\tilde{R}$  is another ramification curve of  $f$  such that  $\tilde{R} \cap V_{o_i, m} \neq \emptyset$ , then for a point  $q \in f(\tilde{R})$  the image  $f_*(\gamma_q)$  of a geometric generator  $\gamma_q$  is a transposition in  $\Sigma_d$ .

**Lemma 3.** Let  $f$  and its branch curve  $\overline{B}_1$  satisfy conditions (R<sub>0</sub>) – (R<sub>3</sub>), and let  $H$  be a subgroup of  $H_{\overline{B}_1}$  generated by  $\sigma_1$  and the transpositions belonging to  $H_{\overline{B}_1}$ . Then  $H = \Sigma_{J_H}$ .

*Proof.* Let  $\sigma = \sigma_1 = \varphi_1(\gamma_{q_1})$  where  $\gamma_{p_1}$  is the geometric generator defined above. Then it is easy to see that condition (R<sub>1</sub>) implies that  $H$  and  $\sigma$  satisfy condition (i) from Lemma 1. Similarly, it follows from conditions (R<sub>2</sub>) and (R<sub>3</sub>) that  $H$  and  $\sigma$  satisfy conditions (ii) and (iii) from Lemma 1. Therefore,  $H = \Sigma_{J_H}$ .  $\square$

**Proposition 2.** *Let  $Z_1, \dots, Z_k$  be the irreducible components  $f : Z \rightarrow Y$  be a ramified covering of a simply connected surface  $Y$ . Assume that the branch locus  $\overline{B}$  of  $f$  satisfies conditions  $(R_0) - (R_3)$ . Then the Galois group  $G$  of  $f$  is isomorphic to  $\Sigma_{d_1} \times \dots \times \Sigma_{d_k}$ , where  $d_i = \deg f|_{Z_i}$ .*

*Proof.* The decomposition  $Z = Z_1 \sqcup \dots \sqcup Z_k$  defines a partition  $J = \{I_1, \dots, I_k\}$  of the set  $f^{-1}(p_0)$ . The group  $G$  leaves invariant the partition  $J$  and acts transitively on each  $I_j \subset J$ . Therefore Proposition 2 follows from Lemma 3 and Corollary 2.  $\square$

### 3. PROOF OF THEOREM 1, 2 AND COROLLARY 1

We use notations defined in Introduction and Section 2.

**3.1. Proof of Theorem 2.** Denote by  $\overline{C}_i = \nu^{-1}(C_i)$  the irreducible components of  $\overline{C}$ ,  $1 \leq i \leq k$ .

Obviously,

$$Z \simeq \{(p, l) \in \overline{C} \times \hat{\mathbb{P}}^2 \mid p \in \overline{C}, \nu(p) \in l\}$$

and it is easy to see that

$$Z_i \simeq \{(p, l) \in \overline{C}_i \times \hat{\mathbb{P}}^2 \mid p \in \overline{C}_i, \nu(p) \in l\}$$

are the irreducible components of the surface  $Z$ .

Let  $t$  be a local parameter in a small neighbourhood  $U \subset \overline{C}$  of a point  $p \in \overline{C}$  and let the normalization  $\nu$  be given in  $U$  by

$$x_1 = \phi_1(t), \quad x_2 = \phi_2(t), \quad x_3 = \phi_3(t). \quad (2)$$

If  $(y_1, y_2, y_3)$  are homogeneous coordinates in  $\hat{\mathbb{P}}^2$  dual to the coordinates  $(x_1, x_2, x_3)$  in  $\mathbb{P}^2$ , then the surface  $Z$ , in the neighbourhood  $U \times \hat{\mathbb{P}}^2 \subset \overline{C} \times \hat{\mathbb{P}}^2$ , is given by equation

$$y_1\phi_1(t) + y_2\phi_2(t) + y_3\phi_3(t) = 0.$$

In particular, if  $\nu$  is given by equations (1), that is,

$$\phi_1 = \sum_{i=s_p}^{\infty} a_i t^i, \quad \phi_2 = t^{r_p}, \quad \phi_3 = 1, \quad (3)$$

then  $Z \cap (U \times \hat{\mathbb{P}}^2)$  lies in  $U \times \mathbb{C}^2$ , where  $\mathbb{C}^2 = \{y_1 \neq 0\}$  is the affine plane in  $\hat{\mathbb{P}}^2$ , and  $Z \cap (U \times \mathbb{C}^2)$  is given by equation

$$\sum_{i=s_p}^{\infty} a_i t^i + z_2 t^{r_p} + z_3 = 0, \quad (4)$$

where  $z_i = y_i/y_1$ ,  $i = 2, 3$ . Therefore  $Z$  is a smooth surface and  $(t, z_2)$  are coordinates in  $Z \cap (U \times \mathbb{C}^2)$ .



The restriction of the covering  $f$  to  $Z \cap (U \times \mathbb{C}^2)$ ,

$$f_U : Z \cap (U \times \mathbb{C}^2) \rightarrow \mathbb{C}^2,$$

is the restriction of the projection  $(t, z_2, z_3) \mapsto (z_2, z_3)$ , therefore it is given by

$$\begin{aligned} z_2 &= z_2, \\ z_3 &= -\left(\sum_{i=s_p}^{\infty} a_i t^i + z_2 t^{r_p}\right). \end{aligned} \quad (5)$$

Its Jacobian is equal

$$J(f_U) = -t^{r_p-1} \left( \sum_{i=s_p}^{\infty} i a_i t^{i-r_p} + r_p z_2 \right).$$

Therefore  $f_U$  is ramified along a curve  $R$  given by equation

$$\frac{1}{r_p} \sum_{i=s_p}^{\infty} a_i t^{i-r_p} + z_2 = 0 \quad (6)$$

with multiplicity two and along the fibre  $F_p = \{t = 0\}$  with multiplicity  $r_p$  if  $r_p \geq 2$  and hence  $f(F_p) = L_p \subset \hat{\mathbb{P}}^2$  is a component of the branch locus of  $f$  if  $r_p \geq 2$ . Note also that  $R$  is a section of the ruled surface  $Z \cap (U \times \mathbb{C}^2) \rightarrow \mathbb{C}^2$  and, in addition, it is the unique section contained in the ramification locus. Therefore to show that  $R$  is a germ of the curve  $\tilde{C}$ , we can assume that the image  $\nu(p)$  is a smooth point of  $C$ , that is, we can assume that  $r_p = 1$  and  $\phi_2(t) = t$ . Then  $R$  is given by  $\phi_1'(t) + z_2 = 0$  and the restriction of  $f_U$  to  $R$  is given by

$$y_1 = 1, \quad y_2 = -\phi_1'(t), \quad y_3 = -\phi_1(t) + t\phi_1'(t). \quad (7)$$

Everyone easily check that that equations (7) together with the equations

$$x_1 = \phi_1(t), \quad x_2 = t, \quad x_3 = 1$$

(defining the germ  $\nu(U)$  of  $C$ ) is a parametrization of  $\check{C} \subset I$  over  $\nu(U)$ .

To count the local degree  $\deg_q f_U$  of the covering  $f_U : Z \cap (U \times \mathbb{C}^2) \rightarrow \mathbb{C}^2$  at the point  $q = (0, 0, 0)$ , first of all, note that the curve  $\{z_2 = 0\}$  does not belong to the ramification locus of  $f_U$ , since  $a_{s_p} \neq 0$  in equation (6). Next, let us choose a new parameter  $t_1$  such that  $t_1^{s_p} = \sum_{i=s_p}^{\infty} a_i t^i$ , then  $f_U$  is given by equations of the form

$$\begin{aligned} z_2 &= z_2, \\ z_3 &= -(t_1^{s_p} + z_2 \sum_{i=r_p}^{\infty} c_i t_1^i) \end{aligned} \quad (8)$$

and to count  $\deg_q f_U$ , it suffices to count the number of points belonging to  $f_U^{-1}((z_{2,0}, z_{3,0}))$ , where a point  $(z_{2,0}, z_{3,0}) \in \text{Im} f_U$  is such that  $z_{2,0} = 0$ ,  $z_{3,0} \neq 0$ , and  $z_{3,0}$  is sufficiently close to zero. It follows from equations (8) that this number is equal to  $s_p$ .  $\square$

**3.2. Proof of Theorem 1.** By Theorem 2, the branch locus  $\overline{B}$  of the dualizing covering  $f : Z \rightarrow \hat{\mathbb{P}}^2$  consists of the curve  $\hat{C}$  and the lines  $L_p$  for which  $r_p \geq 2$ , the ramification locus  $\overline{R}$  consists of the curve  $\tilde{C}$  and the fibres  $F_p$  with  $r_p \geq 2$ . Each line  $L_p$  and each fibre  $F_p$  are smooth and  $F_{p_1} \cap F_{p_2} = \emptyset$  for  $p_1 \neq p_2$ . Next, the restriction  $f|_{F_p} : F_p \rightarrow L_p$  of  $f$  to  $F_p$  is an isomorphism and the restriction  $f|_{\tilde{C}} : \tilde{C} \rightarrow \hat{C}$  of  $f$  to  $\tilde{C}$  is a bi-rational map, and  $f$  is ramified along  $\tilde{C}$  with multiplicity two. Therefore  $f_*(\gamma_l)$  is a transposition for any geometric generator  $\gamma_l \in \pi_1(\hat{\mathbb{P}}^2 \setminus \overline{B}, l_0)$ ,  $l \in \hat{C}$ , and hence the dualizing covering  $f$  and its branch locus  $\overline{B}$  satisfy conditions  $(R_0)$  and  $(R_1)$  (see Section 2). Next, if  $F_p \subset \overline{R}$  then the point  $q = F_p \cap \tilde{C}$  belongs to  $f^{-1}(\text{Sing } \overline{B})$  and  $s_p = \deg_q f > r_p$ , that is,  $f$  and its branch locus  $\overline{B}$  satisfy conditions  $(R_2)$  and  $(R_3)$ . Now Theorem 1 follows from Proposition 2.  $\square$

**3.3. Proof of Corollary 1.** First of all, note that if a line  $l \subset \mathbb{P}^2$  is an irreducible component of the curve  $C$  then for each loop  $l_t \subset T_C$  starting at  $l_0$  the monodromy defined by  $l_t$  leaves fixed the point  $l \cap l_0$ .

Let  $C_1, \dots, C_n$  be the irreducible components of  $C$  and let  $\deg C_i = d_i \geq 2$  for  $i = 1, \dots, k$  and  $\deg C_i = 1$  for  $i > k$ . Denote by  $\text{Sing } C$  the set of singular points of  $C$ , by  $L_{\text{Sing}} = \bigcup_{P \in \text{Sing } C} L_P \subset \hat{\mathbb{P}}^2$ , by  $C' = C_1 \cup \dots \cup C_k$ . Let  $f' : Z \rightarrow \hat{\mathbb{P}}^2$

be the dualizing covering for  $C'$  and  $\overline{B}'$  its branch locus (see Theorem 2). Then it is easy to see that  $T_C = \hat{\mathbb{P}}^2 \setminus (\overline{B}' \cup L_{\text{Sing}})$ .

We have  $f'^{-1}(l_0) = \{(p_1, l_0), (p_2, l_0), \dots, (p_d, l_0)\}$ , where  $p_i = \nu^{-1}(P_i)$ ,  $l_0 \cap C' = \{P_1, P_2, \dots, P_d\}$ , and  $P_1, P_2 \in C_1$ .

The embedding  $i : T_C \hookrightarrow \hat{\mathbb{P}}^2 \setminus \overline{B}'$  defines an epimorphism  $i_* : \pi_1(T_C, l_0) \rightarrow \pi_1(\hat{\mathbb{P}}^2 \setminus \overline{B}', l_0)$ . By Theorem 1, there is an element  $\gamma' \in \pi_1(\hat{\mathbb{P}}^2 \setminus \overline{B}', l_0)$  such that  $f'_*(\gamma') = (1, 2) \in \Sigma_d$ , where  $(1, 2)$  is the transposition transposing the points  $(p_1, l_0)$  and  $(p_2, l_0)$ . Let  $l_t$  be a loop representing an element  $\gamma \in \pi_1(T_C, l_0)$  such that  $i_*(\gamma) = \gamma'$ . It is easy to see that the loop  $l_t$  is a desired one.  $\square$

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